# Potentialism

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# Potentialism

Potentialism refers to the situation where one seeks to understand a structure or a collection of structures by means of a family of partial structures.

There are abundant natural instances of potentialism in mathematics.

# Classical potentialism

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# Classical potentialism

The idea of potentialism goes back to Archimedes and the classical debate between actual and potential infinity.

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## Actualist

Yes, the natural numbers 0, 1, 2 and so on, are infinite, and they form an *actually* infinite set, a completed collection, which can be used in further mathematical constructions.

# The essence of potentialism

The essence of the potentialist idea, however, can be de-coupled from this debate about infinity.

Potentialism is really about the situation that arises when one is trying to describe a mathematical structure in terms of a collection of structure approximations.

Many interesting examples arise in the foundations of mathematics and set theory.

# Set-theoretic potentialism

Set-theoretic **potentialism** is the view that the set-theoretic universe itself is never fully completed, but rather unfolds gradually as parts of it increasingly come into existence or become accessible to us.

On this view, the upper or outer reaches of the set-theoretic universe are seen to have a merely potential character.

# Kinds of set-theoretic potentialism

**Height-potentialism (+ width actualism)**

The universe grows taller as new ordinals are formed, but power sets are actual.

**Width-potentialism (+ height-actualism)**

The universe grows wider as one adds new subsets to infinite sets, such as by forcing. But the ordinals are completed.

**Height- and width-potentialism**

The universe can be made both taller and wider.

# Potentialism as model theory

Ultimately, one looks upon potentialism as a part of model theory, analyzing a class of structures with an extension relation.

# Specific potentialist concepts

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**Rank potentialism**

We view the set-theoretic universe $V$ as being approximated by its rank-initial segments $V_\alpha$ for ordinals $\alpha$. This form a collection of partial universes for the universe $V$.

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## Grothendieck-Zermelo potentialism

Consider as approximation universes all the various $V_\kappa$ where $\kappa$ is an inaccessible cardinal (and assume there are unboundedly many). These worlds $V_\kappa$ are the same as what the category theorists call Grothendieck universes.

# More examples

## Forcing potentialism

Consider the collection of all countable models of set theory, viewing a model $M$ as a fragment of its forcing extensions $M[G]$. We make larger worlds by performing more and more forcing.

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## $\Gamma$-forcing potentialism

We consider the forcing potentialism system, but only allowing forcing notions in $\Gamma$. Natural examples include c.c.c. forcing potentialism, proper forcing potentialism, class forcing potentialism and others.

# More examples

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Consider the countable transitive models of set theory $M$ as a collection of worlds, ordered simply by inclusion $M \subseteq N$. More robust if every real is in a CTM.

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Consider the countable transitive models of set theory $M$ as a collection of worlds, ordered simply by inclusion $M \subseteq N$. More robust if every real is in a CTM.

## Top-extensional potentialism

Consider the countable models of set theory $M$ as a collection of worlds, considering $M$ to be a universe fragment of its top-extensions $N$.

# Arithmetic potentialism

There are also numerous different kinds of arithmetic potentialism.

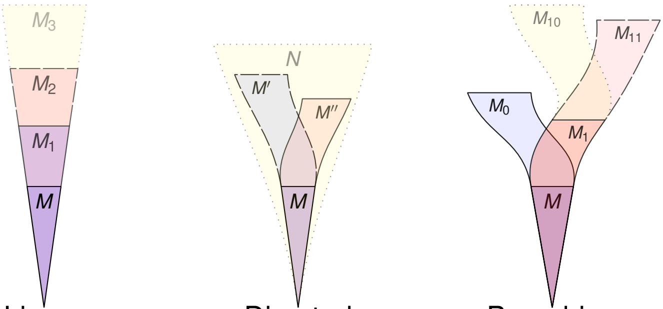
### Arithmetic potentialism

Consider the collection of models of arithmetic PA, under any of various natural extension concepts: end-extension, arbitrary extension, conservative extension, $\Sigma_n$-elementary extension.

# Algebraic structure potentialism

For essentially any kind of algebraic structure, we have corresponding versions of potentialism, in which we study how that structure of class of structures arise from their algebraic fragments.

# Different kinds of potentialism




Linear inevitability
S4.3

Directed convergence
S4.2

Branching possibility
S4

# Model-theoretic account of potentialism

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- A collection $\mathcal{W}$ of structures $M$ in a common language $\mathcal{L}$

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- a reflexive transitive relation on these structures $\sqsubseteq$
- whenever $U \sqsubseteq W$, then $U$ is a substructure of $W$.

Any such collection of models with $\sqsubseteq$ forms a potentialist system.

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$M \models \Diamond\varphi$ if some $N$ with $M \sqsubseteq N$ has $N \models \varphi$.

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**Possibility: $\varphi$ is possible over $M$**

$M \models \Diamond\varphi$ if some $N$ with $M \sqsubseteq N$ has $N \models \varphi$.

**Necessity: $\varphi$ is necessary over $M$**

$M \models \Box\varphi$ if all such $N$ have $N \models \varphi$.

With this modal language, one can often express sweeping general principles describing how truth varies and propagates through the models as one moves upward in the system.

# Semantics of potentialism

Suppose $\mathcal{W}$ is a potentialist system of $\mathcal{L}$-structures.

Language $\mathcal{L}^\Diamond$ augments $\mathcal{L}$ with modal operators $\Diamond, \Box$.

Define the satisfaction relation using the Kripke/Tarski semantics

$$W \models_{\mathcal{W}} \varphi(a)$$

- Atomic, Boolean combinations $\varphi$ are defined as by Tarski.
- Quantifiers are interpreted in the current world $W$.
  $W \models_{\mathcal{W}} \exists x\, \varphi(x, a)$ means
  there is some $x \in W$ with $W \models_{\mathcal{W}} \varphi(x, a)$.
- Modal operators use the $\sqsubseteq$ accessibility relation.
  $\Diamond \varphi$ means $\varphi$ is true in some accessible world
  $\Box \varphi$ means $\varphi$ is true in all accessible worlds.

# Project goals

- To provide precise accounts of the various kinds of potentialism.
- To investigate the modal validities of the various potentialist perspectives.
- To focus on strong natural modal assertions as axioms or hypotheses for the structures

# Easy validities

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If an implication holds in all extensions, and the hypothesis holds in all extensions, then so does the conclusion.

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If an implication holds in all extensions, and the hypothesis holds in all extensions, then so does the conclusion.

**Dual** $\Diamond\varphi \longleftrightarrow \neg\square\neg\varphi$

A statement $\varphi$ holds in some extension, just in case not all extensions have $\neg\varphi$.

# Verifying easy validities

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If $\varphi$ holds in all extensions, then so does $\Box\varphi$, since an extension of an extension is an extension, as $\sqsubseteq$ is transitive.

The modal theory S4 is obtained by closing these axioms under modus ponens and necessitation.

**Conclusion**

S4 is valid in every potentialist system.

# Potentialist validities

## Theorem

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**Theorem**

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**Proof.**

If $\forall x\,\psi(x)$ is true in all worlds accessible from $W$, then for any $x \in W$, we must have $\psi(x)$ in all further worlds, since this $x$ still exists in those worlds. □

# Directed systems

## Theorem

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## Proof.

If $M \models \Diamond\Box\varphi$, then there is $M \sqsubseteq N$ with $N \models \Box\varphi$. Consider any extension $M \sqsubseteq W$. By directedness, there is $U$ with $N, W \sqsubseteq U$, and since $N \models \Box\varphi$, we see $U \models \varphi$, and so $W \models \Diamond\varphi$, and so $M \models \Box\Diamond\varphi$, as desired. □

# Linear systems

## Theorem

*If a potentialist system $\mathcal{W}$ is linearly ordered, then axiom .3*

$$(\Diamond\varphi \wedge \Diamond\psi) \to \Diamond(\varphi \wedge \Diamond\psi) \vee \Diamond(\psi \wedge \Diamond\varphi)$$

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*is valid.*

## Proof.

If world $M$ satisfies $\Diamond\varphi$ and $\Diamond\psi$, then there is an extension of $M$ satisfying $\varphi$ and another extension satisfying $\psi$. By linearity, one of them must have occurred before the other, and that one will fulfill the conclusion of the implication. $\square$

# Summary lower bounds

## Conclusions

- Every potentialist system validates S4.
- Directed systems validate S4.2.
- Linear systems validate S4.3.

# Maximality principle

The *maximality principle* is true at a world $M$ in a potentialist system $\mathcal{W}$ if

$$M \models \Diamond\Box\varphi \to \varphi$$

for every assertion $\varphi$.

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**Goal**

For each of the natural potentialist systems, characterize the models $M$ that fulfill the maximality principle.

# Seeking further validities

M $\quad \Box\Diamond\varphi \implies \Diamond\Box\varphi$

**Valid for Grothendieck-Zermelo potentialism?**

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## Valid for forcing potentialism?

No. $\Box\Diamond\mathrm{CH}$ is true for forcing potentialism in any model of set theory, but $\Diamond\Box\mathrm{CH}$ is never true.

# Consider some examples

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### Valid for top-extensional potentialism?

No. $\Box\Diamond$(eventual $\mathrm{GCH}$) is true for top-extensional potentialism over any countable model of set theory, but $\Diamond\Box$(eventual $\mathrm{GCH}$) is never true.

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Not in every model. Note that $L \models \Diamond\Box V \neq L$ for forcing potentialism. But $L$ does not satisfy $V \neq L$. (Meanwhile, some models of set theory can validate $\Diamond\Box\varphi \to \varphi$ for sentences $\varphi$ or even $\varphi(x)$ for real parameters $x$.)

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**Valid for Grothendieck-Zermelo potentialism?**

Not in every model. Every model $V_\kappa$ satisfies

$\Diamond\Box$*(there are at least five inaccessible cardinals),*

but the smallest models do not yet have five inaccessible cardinals.

**W5** $\Diamond\Box\varphi \implies (\varphi \implies \Box\varphi)$

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$\varphi =$ *either there are an even finite number of inaccessible cardinals or infinitely many.*

So $\Diamond\Box\varphi$ in every $V_\kappa$, and $\varphi$ holds in some small $V_\kappa$, without $\Box\varphi$ yet being true.

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Not in all models. Let

$\varphi = \mathrm{CH}$ *or* $\omega_1^L$ *is collapsed.*

So $\Diamond\Box\varphi$ is true in any model, since we can force to collapse $\omega_1^L$, but $\varphi$ can be true because of $\mathrm{CH}$, without yet having $\Box\varphi$.

# Axiom .3 in forcing potentialism

$$.3 \qquad (\Diamond\varphi \wedge \Diamond\psi) \to \Diamond(\varphi \wedge \Diamond\psi) \vee \Diamond(\psi \wedge \Diamond\varphi)$$

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## Valid for forcing potentialism?

Not in all models. Start in $L$ and let $\omega_1 = S \sqcup T$ be the $L$-least partition into disjoint stationary sets. Let $\varphi$ assert that $S$ is stationary but $T$ is not, and let $\psi$ assert $T$ is stationary and $S$ is not. Both of these are forceable over $L$, but once one of them is true, the other one becomes impossible to force.

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Meanwhile, (.2) is valid for c.c.c. forcing in any model of $\mathrm{MA}$.

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Meanwhile, (.2) is valid for c.c.c. forcing in any model of $\mathrm{MA}$.

In this way, the modal language can express fundamental principles of forcing and how it relates to truth.

I view a valid modal principle for forcing as a kind of forcing

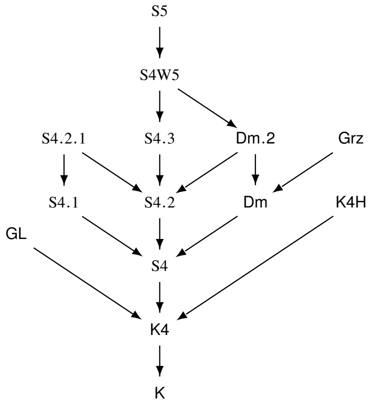
# More axioms beyond S4.2

$$
\begin{array}{rl}
5 & \Diamond\Box\varphi \implies \varphi \\
\mathrm{M} & \Box\Diamond\varphi \implies \Diamond\Box\varphi \\
\mathrm{W5} & \Diamond\Box\varphi \implies (\varphi \implies \Box\varphi) \\
.3 & \Diamond\varphi \wedge \Diamond\psi \implies (\Diamond(\varphi \wedge \Diamond\psi) \vee \Diamond(\varphi \wedge \psi) \vee \Diamond(\psi \wedge \Diamond\varphi)) \\
\mathrm{Dm} & \Box(\Box(\varphi \implies \Box\varphi) \implies \varphi) \implies (\Diamond\Box\varphi \implies \varphi) \\
\mathrm{Grz} & \Box(\Box(\varphi \implies \Box\varphi) \implies \varphi) \implies \varphi \\
\text{Löb} & \Box(\Box\varphi \implies \varphi) \implies \Box\varphi \\
\mathrm{H} & \varphi \implies \Box(\Diamond\varphi \implies \varphi)
\end{array}
$$

It is a fun exercise to consider them in the various potentialist systems.

# Some common modal theories

| | | |
|---|---|---|
| S5 | = | S4 + 5 |
| S4W5 | = | S4 + W5 |
| S4.3 | = | S4 + .3 |
| S4.2.1 | = | S4 + .2 + M |
| S4.2 | = | S4 + .2 |
| S4.1 | = | S4 + M |
| S4 | = | K4 + S |
| Dm.2 | = | S4.2 + Dm |
| Dm | = | S4 + Dm |
| Grz | = | K + Grz |
| GL | = | K4 + Löb |
| K4H | = | K4 + H |
| K4 | = | K + 4 |
| K | = | K + Dual |



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- The maximality principle $\Diamond\Box\varphi \to \varphi$ is S5.
- We showed various non-validities with ad-hoc counterexamples.

# Finding the exact modal validities

We can often easily observe that a specific modal theory, such as S4, S4.2 or S4.3 is valid in a given potentialist system. In this sense, lower bounds are usually easy.

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It is more difficult and subtle to prove upper bounds.

One can often refute the validity of individual assertions in an ad hoc manner, as we just did in a few cases.

But in order to identify the exact modal theory of validities, we need more powerful tools.

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It turns out that we can establish upper bounds on the validities of a potentialist system by identifying certain kinds of control statements for the models in the system:

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- Switches
- Dials
- Buttons
- Ratchets
- Railyards

So let me explain what these statements are and how they connect to the modal theories.

# Potentialist Validities

A modal assertion $\varphi(p_0, \ldots, p_n)$ is *valid* at world $W$ in potentialist system $\mathcal{W}$ if

$$W \models_{\mathcal{W}} \varphi(\psi_0, \ldots, \psi_n)$$

for all assertions $\psi_i$ from $\mathcal{L}^\diamond$ (or sometimes $\mathcal{L}$, possibly parameters from $W$ allowed).

Each validity is really a scheme of truth assertions.

In some cases, it matters whether one considers only $\mathcal{L}$-instances or $\mathcal{L}^\diamond$ or whether parameters are allowed.

# Switches

A *switch* in a potentialist system is a statement $s$ that can always be turned on or off by accessing another world.

Thus, $\Diamond s$ and $\Diamond \neg s$ are true at every world.

A family of switches $s_0, \ldots, s_n$ is *independent*, if every world can access a world realizing any given finite truth pattern.

# Switches in forcing potentialism

The continuum hypothesis $\mathrm{CH}$ is a switch with respect to forcing potentialism, since $\mathrm{CH}$ and $\neg\mathrm{CH}$ are both forceable over any model of set theory.

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We can make an independent family of switches

$$s_n = \mathrm{GCH} \text{ holds at } \aleph_n.$$

The truth values of these statements can be made to conform with any desired pattern.

# Switches $\to$ S5

**Theorem**

*If a potentialist system $\mathcal{W}$ has arbitrarily large families of independent switches, then the validities of each world are within S5.*

**Proof.**

If $\varphi$ is not in S5, then it fails in a propositional Kripke model $M$ with a finite frame in which every world accesses all others. Associate each world $w$ in $M$ with a switch pattern $\Phi_w$. For each propositional variable $p$, let

$$\psi_p = \bigvee\{\, \Phi_w \mid p \text{ is true in } w \,\}.$$

$M$ is simulated inside $\mathcal{W}$ via

$$W \models_{\mathcal{W}} \phi(\psi_{p_0}, \ldots, \psi_{p_n}) \qquad \longleftrightarrow \qquad (M, w) \models \phi(p_0, \ldots, p_n),$$

when $W$ satisfies $\Phi_w$. This instance shows $\varphi$ is not valid in $\mathcal{W}$. □

# Dials

A *dial* is a list of statements $d_0, d_1, d_2, \ldots$, such that every world in $\mathcal{W}$ satisfies exactly one of them, and each is possible from any world.

**Theorem**

*A potentialist system has arbitrarily large independent switches iff it has arbitrarily large finite dials.*

Each dial $d_r$ asserts a switch pattern. Each switch asserts a binary digit of the dial index $d_r$.

**Example dials in forcing potentialism**

Let $d_n$ be true if the continuum is $\aleph_{n+1}$.

# Buttons

A *button* is a statement $b$ such that $\Diamond\Box b$ is true at every world.

The button is *pushed* if $\Box b$, and otherwise unpushed.

A *pure* button is one for which $\Box(b \to \Box b)$.

A family of buttons and switches is *independent* if you can control them as desired: push any button without pushing others, and set the switches as desired.

# Buttons in forcing

Collapsing $\omega_1^L$ is a button with respect to forcing potentialism.

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But it is actually quite subtle to get independent buttons this way.

Instead, start in $L$ and partition $\omega_1 = \bigsqcup_n S_n$ into $L$-least stationary partition. Let $b_n = S_n$ is not stationary. These are independent buttons, because we can always shoot a club so as to destroy a specific stationary set, while preserving stationarity of all stationary sets in the complement.

# Buttons in models of arithmetic

In the potentialist system of the models of $\mathrm{PA}$ under end-extension, the assertion $\neg\mathrm{Con}(\mathrm{PA})$ is a button, since once true, it remains true in all further extensions, and in any model in which $\mathrm{Con}(\mathrm{PA})$ holds, one can form the Henkin theory of $\mathrm{PA} + \neg\mathrm{Con}(\mathrm{PA})$. The corresponding Henkin model is definable and so forms an end-extension of the given model.

So $\neg\mathrm{Con}(\mathrm{PA})$ is a button.

# Buttons → S4.2

**Theorem**

*If a potentialist system $\mathcal{W}$ has (arbitrarily many) independent buttons and switches (or buttons and a dial), then the validities of any world where the buttons are not yet pushed are contained within S4.2.*

**Proof.**

As in the modal logic of forcing (Hamkins,Löwe).

Using buttons and switches, can simulate any Kripke model built on a finite pre-Boolean algebra frame, which is complete for S4.2. □

# Ratchets

A *ratchet* is a sequence of buttons $r_1, \ldots, r_n$, such that each implies all the earlier, and each can be pushed without pushing the next.

So a ratchet has one-way operation: the ratchet volume can only go up.

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So a ratchet has one-way operation: the ratchet volume can only go up.

**Example ratchet in c.c.c. forcing potentialism**

Let $r_n$ assert that $2^\omega \geq \aleph_n$.

# Ratchets → S4.3

**Theorem**

*If a world in Kripke model $\mathcal{W}$ has arbitrarily large ratchets + independent switches (or a dial), then the validities are within S4.3.*

The proof similarly is to simulate the Kripke models with finite linear pre-order frames inside $\mathcal{W}$.

# Long ratchets

In a model of set theory, a *long ratchet* is a formula $\varphi(\alpha)$ with ordinal parameter $\alpha$, which form a ratchet.

With a long ratchet, we don't need the independent switches, since we can simulate them by the position within an $\omega$-block.

So any model of set theory with a long ratchet has its validities contained within S4.3.

# Railyards

A *railway switch* is a statement $r$ such that $\Diamond\Box r$ and $\Diamond\Box\neg r$ holds at a world where it is not yet switched.

The railway train has passed when $\Box r$ or $\Box\neg r$. It is too late to switch to the other track.

# Railyards

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The railway train has passed when $\Box r$ or $\Box\neg r$. It is too late to switch to the other track.

Note that this contradicts S4.2, since at a world where $r$ is not yet switched, we cannot fulfill axiom (.2) $\Diamond\Box r \to \Box\Diamond r$.

# Example railway switch in models of arithmetic

The Rosser sentence $\rho$, which asserts that for any proof of $\rho$, there is a shorter proof of $\neg\rho$.

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What is at stake, in models where they are provable, is which statement has the shorter proof.

This sentence is a railway switch in the standard model, since neither $\rho$ nor $\neg\rho$ is provable there, but we can find some extensions where $\rho$ has the shorter proof and other extensions where $\neg\rho$ has the shorter proof.

So $\Diamond\Box\rho$ and $\Diamond\Box\neg\rho$ both hold in the standard model, with respect to end-extensional arithmetic potentialism.

# Railway switch for c.c.c. forcing

Consider the potentialist system of c.c.c. forcing extensions.

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Let $r$ be the statement, "$T$ is special". This is c.c.c. forceable in $L$, and once it becomes special, then it remains special. So $\Diamond\Box r$.

# Railway switch for c.c.c. forcing

Consider the potentialist system of c.c.c. forcing extensions.

Start in $L$. Let $T$ be the $L$-least Suslin tree.

Let $r$ be the statement, "$T$ is special". This is c.c.c. forceable in $L$, and once it becomes special, then it remains special. So $\Diamond\Box r$.

But since we could also c.c.c. force to add a branch to $T$, and this would prevent the tree from ever becoming special in a c.c.c extension, we have $\Diamond\Box\neg r$.

# Railyards → S4

A railyard is an assemblage of railway switches.

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Specifically, for any finite pretree $T$, we associate to each node $t$ a statement $r_t$ such that every world in $\mathcal{W}$ satisfies exactly one of the statements $r_t$, and if a world $W$ satisfies $r_t$, then it satisfies $\Diamond r_s$ just in case $t \leq s$ in the tree.

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Specifically, for any finite pretree $T$, we associate to each node $t$ a statement $r_t$ such that every world in $\mathcal{W}$ satisfies exactly one of the statements $r_t$, and if a world $W$ satisfies $r_t$, then it satisfies $\Diamond r_s$ just in case $t \leq s$ in the tree.

In other words, the $T$-railyard labeling partitions the worlds of $\mathcal{W}$ into classes, from which possibility looks just like $T$.

**Theorem**

*If $\mathcal{W}$ admits a $T$-railyard labeling for every finite pre-tree $T$, then the modal validities of $\mathcal{W}$ are exactly S4.*

# Summary of the control statement method

In order to find upper bounds on the potentialist validities of a given potentialist system, one should try to identify the various kinds of control statements in the system.

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In order to find upper bounds on the potentialist validities of a given potentialist system, one should try to identify the various kinds of control statements in the system.

Each kind of control statement provides an upper bound on the validities of the system.

In many cases, these upper bounds line up with the easy lower bounds, and one has identified exactly the validities of the system.

# Control statement upper bounds

Consider a potentialist system $\mathcal{W}$.

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# Control statement upper bounds

Consider a potentialist system $\mathcal{W}$.

- If independent switches, then validities $\subseteq$ S5.
- If independent switches+buttons, then validities $\subseteq$ S4.2.
- If switches+ratchet or just long ratchet, then validities $\subseteq$ S4.3.
- If railyard labelings, then validities $\subseteq$ S4.

In each case, the validities are with respect to the language in which the control statements are made.

# A major advantage of control statements

Finding the various kinds of control statements often involves expertise only in the theory of the models themselves, rather than special knowledge of modal logic.

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Finding the various kinds of control statements often involves expertise only in the theory of the models themselves, rather than special knowledge of modal logic.

Set theorists can find their potentialist validities using only set-theoretic ideas.

Model theorists can find their potentialist validities only only model theory.

In the next section, we'll illustrate how this plays out for the various potentialist systems we are considering.

# Coherent potentialist systems

A potentialist system $\mathcal{W}$ is *convergent*, with limit $M$, if

- Every world in $\mathcal{W}$ is a substructure of $M$.
- Every world in $\mathcal{W}$ can be extended so as to accommodate any desired individual of $M$.

This is a weak form of directedness.

Examples:

- finite (or finitely generated) substructures of a given structure.
- countable substructures of a fixed uncountable structure.

# The Potentialist translation

For every $\psi$ in $\mathcal{L}$, form the *potentialist translation* $\psi^\Diamond$ by

replace $\exists x$ with $\Diamond\exists x$; replace $\forall x$ with $\Box\forall x$.

**Theorem**

*If potentialist system $\mathcal{W}$ has limit $M$, then*

$$M \models \psi(a) \qquad \longleftrightarrow \qquad W \models_{\mathcal{W}} \psi^\Diamond(a),$$

*for any world $W \in \mathcal{W}$ in which the individual $a$ exists.*

Thus, actual truth in the limit structure amounts to potentialist truth in the approximating structures. So the potentialist can in effect refer to actual truth.

Proved by simple induction on formulas.

# Rank-potentialism

First, consider set-theoretic rank-potentialism.

Rank-potentialism arises from the potentialist system consisting of the sets $V_\beta$, the rank-initial segments of the cumulative hiearchy.

In this system, $\Diamond\varphi$ is true at some $V_\beta$, if there is a larger $V_\delta$ in which $\varphi$ is true.

# Modal validities of rank-potentialism

**Theorem**

*For set-theoretic rank-potentialism,*

1. *Every S4.3 assertion is valid in every $V_\beta$ for any $\mathcal{L}_\in^\Diamond$ assertion with parameters from $V_\beta$.*
2. *Some worlds validate only the S4.3 assertions.*
3. *Validities at any world are within S5.*

**Proof.**

The $V_\beta$ are linearly ordered, so S4.3 is valid.

Long ratchet: "$\aleph_\alpha$ exists." □

# The potentialist maximality principle

Meanwhile, some $V_\delta$ can exhibit additional validities.

5 $\quad \Diamond\Box\varphi \to \varphi$

**Theorem**

*The following are equivalent for any ordinal $\delta$:*

1. *$V_\delta$ satisfies the maximality principle S5 for $\mathcal{L}_\in$-assertions with parameters.*
2. *$\delta$ is $\Sigma_3$-correct. That is, $V_\delta \prec_{\Sigma_3} V$.*

**Proof.**

(2 $\implies$ 1) Assume $\delta$ is $\Sigma_3$-correct and $V_\delta \models \Diamond\Box\varphi(a)$. So $\exists\lambda{\geq}\delta\ \forall\theta{\geq}\lambda\ V_\theta \models \varphi(a)$. This is $\Sigma_3$. It follows that $V_\delta \models \varphi(a)$.

(1 $\implies$ 2) If S5 is valid at $V_\delta$, then $\delta = \beth_\delta$. If $\exists x \forall\beta\ V_\beta \models \varphi(a)$, then $V_\delta \models \Diamond\Box\exists x\forall\beta\ V_\beta \models \varphi(a)$. By S5, it is true in $V_\delta$. □

# The language matters

Allowing assertions from the potentialist language is strictly stronger.

**Theorem**

*The following schemes are equivalent:*

1. $V_\delta$ *satisfies the maximality principle S5 with respect to* $\mathcal{L}_\in^\Diamond$*-assertions with parameters.*
2. $\delta$ *is a correct cardinal,* $V_\delta \prec V$.

The point is that the modal operators $\Diamond$ and $\exists$ in $V_\delta$ work essentially as quantifiers in $V$, by the potentialist translation.

ZFC proves $\Sigma_3$-correct cardinals exist, but it doesn't prove that fully correct cardinals exist.

# Variations on rank-potentialism

One can refine the potentialist system by allowing only certain $V_\beta$, for $\beta$ in some class $A$.

These are still linearly ordered, so S4.3 remains valid.

And one can still make a long ratchet: "there are at least $\alpha$ many elements in $A$." So some worlds have exactly S4.3.

S5 is valid at $V_\delta$ iff $\delta$ is $\Sigma_3(A)$-correct.

# Grothendieck-Zermelo universes

The potentialist perspective is well illustrated in current mathematical practice by the use of Grothendieck-Zermelo universes in category theory: $V_\kappa$ for inaccessible cardinal $\kappa$.

Category-theorists use these universes in a potentialist manner. Work inside one universe $V_\kappa$, but if needed, move to a higher one.

Zermelo also had this perspective explicitly (1930).

> *What appears as an 'ultrafinite non- or super-set' [a proper class] in one model is, in the succeeding model, a perfectly good, valid set with both a cardinal number and an ordinal type, and is itself a foundation stone for the construction of a new domain.*

# Grothendieck-Zermelo potentialism

Assume the Grothendieck universe axiom. Then:

- S4.3 is valid at every GZ-universe $V_\kappa$.

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Assume the Grothendieck universe axiom. Then:

- S4.3 is valid at every GZ-universe $V_\kappa$.
- Some GZ-universes have only S4.3 as valid.
- S5 is valid at GZ-universe $V_\kappa$, for $\mathcal{L}_\in$-assertions with parameters, if and only if $\kappa$ is $\Sigma_3$-reflecting.

# Grothendieck-Zermelo potentialism

Assume the Grothendieck universe axiom. Then:

- S4.3 is valid at every GZ-universe $V_\kappa$.
- Some GZ-universes have only S4.3 as valid.
- S5 is valid at GZ-universe $V_\kappa$, for $\mathcal{L}_\in$-assertions with parameters, if and only if $\kappa$ is $\Sigma_3$-reflecting.
- S5 is valid at $V_\kappa$, for $\mathcal{L}_\in^\Diamond$-assertions with parameters, iff $\kappa$ is fully reflecting.

# Transitive-set potentialism

Consider next the potentialist system of all transitive sets

$$\mathcal{T} = \{\, W \mid W \text{ is transitive} \,\}.$$

So $\Diamond\psi$ is true at $W$ if there is a larger transitive set with $\psi$.

This system exhibits potentialism both with respect to height and width.

But width can eventually stabilize. For example, every set $x$ eventually gets its full power set, containing not only all subsets, but all potential subsets.

$$\forall x\ \Diamond\exists y\ \Box\, y = P(x)$$

# Modal logic of transitive set potentialism

## Theorem

*The propositional modal validities of transitive-set-potentialism are exactly the assertions of S4.2.*

1. *S4.2 is valid in every world, for assertions in $\mathcal{L}_{\in}^{\Diamond}$ with parameters.*
2. *Some worlds validate only S4.2.*
3. *For any particular world, validities are within S5.*

## Proof.

Upward directed, so S4.2 is valid.

Provide independent buttons and switches to get exactly S4.2. □

# Maximality principle S5

## Theorem

*The following are equivalent in transitive-set potentialism.*

1. *S5 is valid at M for $\mathcal{L}_\in$-assertions with parameters.*
2. *$M = V_\delta$, for some $\Sigma_2$-correct cardinal $\delta$.*

## Proof.

(2 $\implies$ 1) Suppose $\delta$ is $\Sigma_2$-correct, and assume $\Diamond\Box\varphi(a)$ holds at $V_\delta$. So there is transitive set $N \supseteq V_\delta$ with all $U \supseteq N$ having $\varphi(a)$. This is $\Sigma_2$. So already such $N$ inside $V_\delta$. So $V_\delta \models \varphi(a)$.

(1 $\implies$ 2) Assume S5 at $M$. Show $M$ is correct about power sets. Similar argument shows $M = V_\delta$ some $\delta$. Use $\Diamond\Box\varphi(a) \to \varphi(a)$ to conclude $\delta$ is $\Sigma_2$-correct. □

# Strong maximality principle

If you want S5 for assertions in the potentialist language $\mathcal{L}_\in^\Diamond$, then it is stronger.

## Theorem

*The following are equivalent in transitive-set potentialism.*

1. *S5 is valid at world M for $\mathcal{L}_\in^\Diamond$-assertions with parameters.*
2. $M \prec V$*. In other words,* $M = V_\delta$ *for a correct cardinal* $\delta$*.*

# Variations on transitive set potentialism

It is natural to want only transitive models of a particular nice theory $T$.

Consider this as a potentialist system, and assume every $x \in V$ is an element of such a model. Then:

S4.2 is valid at every world, for $\mathcal{L}_{\in}^{\Diamond}$-assertions with parameters.

Examples show that some worlds can exhibit exactly S4.2, or exactly S4.3, or exactly some intermediate theory, depending on the theory $T$ and the set-theoretic background.

# Solovay's modalities

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It might seem at first that this is the same thing we are doing with potentialism.

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Solovay had studied the modalities of “true in all transitive sets” and “true in all $V_\kappa$ for inaccessible $\kappa$.”

It might seem at first that this is the same thing we are doing with potentialism.

But it is not the same.

Solovay's modalities are not potentialist, since in effect they are oriented downward, rather than upward. For Solovay, $\Diamond\Box\varphi$ is true at $V_\kappa$ if there is a *smaller* $V_\beta$ such that $\varphi$ is true inside all still smaller $V_\delta$.

In contrast, potentialism is upward oriented.

# CTM potentialism

Consider the collection of countable transitive models of ZFC as a potentialist system.

This system exhibits both height- and width-potentialism.

# CTM potentialist validities

Assume every real is in a countable transitive model of ZFC (a weak large cardinal axiom). Then:

- Collection of countable transitive models of ZFC provides a potentialist account of $H_{\omega_1}$.
- S4.2 is valid at every world, any language, with parameters.
- Some worlds validate only S4.2.
- Validities are always within S5.
- Some worlds validate S5 for sentences, no parameters.

For 3, use the Shephardson-Cohen model, which has buttons and switches.

With parameters, all worlds have exactly S4.2 being valid.

# Maximality principle for CTM potentialism

Let's elucidate the validity of S5, the maximality principle, for CTM potentialism.

$$\Diamond\Box\varphi \to \varphi$$

### Theorem

*If every real is in a countable transitive model of* $\mathrm{ZFC}$*, then every world* $U \in \mathcal{C}$ *can be extended to a world* $W \in \mathcal{C}$ *validating S5 in any countable language extending* $\mathcal{L}_{\in}^{\Diamond}$ *(interpreted in every model of* $\mathcal{C}$*) with real parameters from* $U$*.*

### Proof.

The CTMs are upward $\sigma$-closed. Countably many instances of S5 to fulfill. Build a tower of models, achieving $\Diamond\Box\varphi_n$ at stage $n$, if possible. Any model above the tower has S5. □

# V=L and maximize

Although $V = L$ is often viewed as limiting, nevertheless in the potentialist system of CTMs, it is possible that $V = L$ is always recoverable by moving to a taller model, even when there are CTMs satisfying ZFC plus many large cardinals.

This perspective undercuts the view of $V = L$ as necessarily limiting.

# Countable models of ZFC

Consider the potentialist system consisting of the countable models of ZFC, under the substructure relation.

This includes the nonstandard models of set theory.

# Countable models of set theory

**Theorem**

*Consider the potentialist system of all countable models of* ZFC*, under the substructure relation.*

1. *S4.3 is valid at every world $W$ for $\mathcal{L}_\in^\Diamond$ assertions using parameters from $W$.*
2. *The validities of any particular world $W$ are contained with S4.3, when restricted to $\mathcal{L}_\in$-assertions with parameters.*
3. *The validities of any particular world are contained within S5, when restricted to sentences in the language of set theory.*
4. *S5 is valid at every countable nonstandard model $W$ of* ZFC *for $\mathcal{L}_\in^\Diamond$ sentences.*

S4.3 follows from my embedding theorem: the countable models of set theory are linearly pre-ordered by embeddability.

# Modal logic of forcing

Consider next the set-theoretic universe $V$ in the potentialist context of all its forcing extensions.

This is width-potentialism, height-actualism.

Benedikt Löwe and I studied the modal validities that arise in this system.

# Modal logic of forcing

**Theorem (Hamkins,Löwe)**

*In the potentialist system of all forcing extensions of a fixed countable model of* ZFC*,*

1. *Exactly S4.2 is valid at every world, for $\mathcal{L}_\in^\Diamond$-assertions with parameters.*
2. *The validities of any particular world are within S5.*
3. *Some models have exactly S4.2 as their set of validities.*
4. *Depending on the original model, some models have S5 valid for sentences.*

# S4.2 is valid for forcing potentialism

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Indeed, the generic multiverse of a countable transitive model of set theory is not directed. For any $M$, there are $M$-generic Cohen reals $c$ and $d$, such that $M[c]$ and $M[d]$ are non-amalgamable, having no common extension to a model with the same ordinals.

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Indeed, the generic multiverse of a countable transitive model of set theory is not directed. For any $M$, there are $M$-generic Cohen reals $c$ and $d$, such that $M[c]$ and $M[d]$ are non-amalgamable, having no common extension to a model with the same ordinals.

Can verify S4.2 nevertheless. If $\varphi$ is forceably necessary over $M$, then fix $M[G] \models \Box\varphi$. For any other forcing notion $\mathbb{Q}$, consider $M[G]$-generic filter $H \subseteq \mathbb{Q}$. Since $M[H]$ is amalgamable with $M[G]$, it follows that $\varphi$ holds in some extension of $M[H]$, and so $\varphi$ is neccessarily forceable in $M$. So (.2) holds.

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Fix a partition of $\omega_1 = \bigsqcup_n S_n$ into disjoint stationary sets. Let $b_n$ assert that $S_n$ is not stationary. These are independent buttons.

So the modal logic of forcing is exactly S4.2, with parameters.

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So the modal logic of forcing is exactly S4.2, with parameters.

In $L$, no parameters are needed, since the partition is definable.

# Maximality principle for forcing

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Asserts any forceably necessary statement is already true.

Introduced by Stavi/Väänänen and independently by Hamkins, who introduced the forcing modalities.

In general, cannot allow uncountable parameters, since

$$\Diamond\Box(x \text{ is countable})$$

is true for any particular set $x$.

# Consistency of the maximality principle

Two proof methods for consistency.

- Compactness argument. Can safely add any finitely many instances of MP to ZFC.

# Consistency of the maximality principle

Two proof methods for consistency.

- Compactness argument. Can safely add any finitely many instances of MP to ZFC.
- Forcing iteration. Requires a fully reflecting cardinal $V_\kappa \prec V$.

The forcing iteration method can accommodate real parameters, if $\kappa$ is inaccessible.

# Modal logic of Γ-forcing

There are many open questions concerning the modal logic of Γ-forcing for natural classes of Γ.

- c.c.c. forcing
- proper forcing
- semi-proper
- class forcing
- many others

In these cases, we know S4.2 is not valid.

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In these cases, we know S4.2 is not valid.

For example, we showed earlier that c.c.c. forcing does not validate (.2) in $L$, since we could specialize the $L$-least tree or force a branch, so this is a railway switch, which violates (.2).

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- many others

In these cases, we know S4.2 is not valid.

For example, we showed earlier that c.c.c. forcing does not validate (.2) in $L$, since we could specialize the $L$-least tree or force a branch, so this is a railway switch, which violates (.2).

Please help solve the modal logic of c.c.c. forcing and others.

# Generic multiverse potentialism

The *generic multiverse* of a model $M$ of set theory is obtained by closing under the operations of forcing extension and ground.

This forms a natural potentialist system.

The modal validities are identical to that in the modal logic of forcing.

# Generic multiverse rank-potentialism

It is interesting to combine rank-potentialism with generic-multiverse-potentialism.

Consider a model of set theory $M$ in the context of its generic multiverse. Form the potentialist system of all $V_\beta^W$, where $W$ is in the generic multiverse of $M$.

So this is height-and-width-potentialism, since we can always force outward, adding more subsets, and we can add more ordinals on top.

# Validities of generic-multiverse rank-potentialism

### Theorem

*For generic-multiverse rank-potentialism over a fixed countable model of* ZFC.

1. *S4.2 is valid at every world for $\mathcal{L}_{\in}^{\Diamond}$-assertions with parameters.*
2. *The validities of any particular world are contained within S5, even when restricted to the sentences of set theory.*
3. *If* ZFC *is consistent, then examples show some worlds validate only S4.2.*

# Arithmetic potentialism

Let us consider the models of PA as forming a potentialist system.

There are a variety of natural extension concepts.

# Natural extension modalities in arithmetic

Consider the potentialist system consisting of the models of PA under top-extensions

$$M \models \Diamond\varphi \quad \longleftrightarrow \quad \varphi \text{ holds in some end-extension of } M, \text{ and}$$
$$M \models \Box\varphi \quad \longleftrightarrow \quad \varphi \text{ holds in all end-extensions of } M.$$

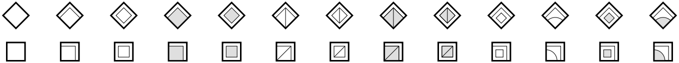
The arbitrary-extension modality, in contrast, is defined by:

$$M \models \Diamond\!\!\!\Diamond\varphi \quad \longleftrightarrow \quad \varphi \text{ holds in some extension of } M$$
$$M \models \boxdot\varphi \quad \longleftrightarrow \quad \varphi \text{ holds in all extensions of } M$$

For generality, consider arbitary c.e. consistent extension $\mathrm{PA}^+$.

# Other natural modalities

Actually, there are many extension concepts for the models of arithmetic.



Top extensions, arbitrary extensions, computably saturated extensions, conservative extensions and combinations.

## Theorem

*In the potentialist system of the models of* $\mathrm{PA}^+$ *under the end-extension modality* $\Diamond$*:*

1. *The potentialist validities of any* $M \models \mathrm{PA}^+$*, with respect to arithmetic assertions with parameters from* $M$ *and indeed one specific parameter suffices, are exactly the modal assertions of* $\mathrm{S4}$*.*
2. *The potentialist validities of any* $M \models \mathrm{PA}^+$*, with respect to arithmetic sentences, is a modal theory containing* $\mathrm{S4}$ *and contained in* $\mathrm{S5}$*.*
3. *Both of the bounds in (2) are sharp: there are models validating exactly* $\mathrm{S4}$ *and others validating exactly* $\mathrm{S5}$*.*

# Exactly S4 wrt language with parameters

To show that exactly S4 is valid for top-extensional arithmetic potentialism, it suffices to find railyards in the potentialist system.

# Exactly S4 wrt language with parameters

To show that exactly S4 is valid for top-extensional arithmetic potentialism, it suffices to find railyards in the potentialist system.

This is a consequence of the universal algorithm.

# The universal algorithm

**Theorem (Woodin)**

*There is a Turing machine program e such that:*

1. *Program e enumerates a finite sequence only, and* PA *proves this.*

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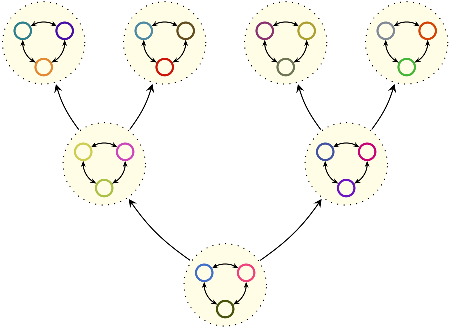
3. *In any model* $M \models \mathrm{PA}$*, if e enumerates sequence s and* $s \subseteq t$ *in M, then there is an end-extension N of M in which e enumerates t.*

In particular, every finite sequence $s \in \mathbb{N}^{<\omega}$ is enumerated by $e$ in some model $M \models \mathrm{PA}$.

I spoke on this theorem at the Beauty of Logic 2018 last week in P...

# Universal algorithm → railyards

We can interpret the numbers on the universal sequence as instructions for how to climb or move around in an any given finite pre-tree. Map each node $t \mapsto r_t$ to a statement about the universal sequence, so that extending the sequence corresponds to climbing in the tree. This is a railyard, and so the modal logic is exactly S4.



# Arithmetic maximality principle

**Theorem**

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Every model of PA has an extension to a model with the maximality principle.

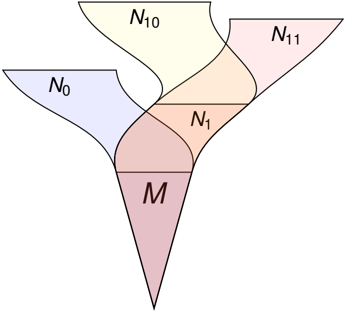
# Top-extensions of models of set theory

Consider now top-extensions for models of set theory.

A model *N* is a *top-extension* (also called *rank-extension*) of *M* if $M \subseteq N$ and all new elements of *N* have rank above the ordinals of *M*. Equivalently, $V_\alpha^M = V_\alpha^N$ for all ordinals $\alpha \in M$.

# Models of set theory under top-extension

Consider the countable models of ZFC under top-extension.



## Theorem

*In the potentialist system consisting of the countable models of* ZFC *under top-extensions:*

1. *Exactly S4 is valid with respect to assertions in* $\mathcal{L}_\in$ *with parameters.*
2. *For any countable* $M \models \mathrm{ZFC}$, *there is parameter* $n \in \omega^M$ *such that exactly S4 is valid with respect to assertions in* $\mathcal{L}_\in(n)$.
3. *For sentences, the validities are between S4 and S5.*
4. *These bounds are sharp; both endpoints are realized.*

# Showing S4 as upper bound

To establish S4 as an upper bound, it suffices to find railyards in top-extensional set-theoretic potentialism.

For this, we try to find a set-theoretic analogue of the universal algorithm.

# Set-theoretic analogue

What is the set-theoretic analogue of the universal algorithm?

One wants a version of the theorem for models of set theory.

# Arithmetic vs. set theory

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The computably enumerable sets are gradually revealed as time proceeds. Elements are confirmed at some stage of time.

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The locally verifiable sets have members confirmed in some $V_\alpha$, as the set-theoretic universe grows.

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## Elementary Fact

Locally verifiable sets $= \Sigma_2$ definable.

# Question

So the set-theoretic analogue of c.e. is $\Sigma_2$ definable.

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## Question(Hamkins)

Is there a $\Sigma_2$ definable set $\{\, x \mid \varphi(x) \,\}$ with the following?

- $\mathrm{ZFC}$ proves $\{\, x \mid \varphi(x) \,\}$ is a set.
- For every countable model $M \models \mathrm{ZFC}$, if

$$M \models \{\, x \mid \varphi(x) \,\} = y \subseteq z,$$

  then there is top-extension $N$ with

$$N \models \{\, x \mid \varphi(x) \,\} = z.$$

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**Theorem (Hamkins + Woodin)**

*There is a $\Sigma_2$ definition $\varphi$ such that*

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*There is a $\Sigma_2$ definition $\varphi$ such that*

1. ZFC *proves* $\{\, x \mid \varphi(x) \,\}$ *is finite.*
2. *If $M$ is transitive, then* $M \models \{\, x \mid \varphi(x) \,\} = \emptyset$.
3. *If $M$ is a countable model of* ZFC *with*

   $$M \models \{\, x \mid \varphi(x) \,\} = y \subseteq z,$$

   *where $z$ is finite in $M$, then there is top-extension $N$ with*

   $$N \models \{\, x \mid \varphi(x) \,\} = z.$$

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The union of the countable members of the universal finite set is an arbitrary countable set.

And so on for other cardinals or other kinds of universal sets.

Also, there is a sequence version of the theorem.

# Validities are S4

**Theorem**

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**Proof.**

Interpret the universal finite sequence as a way to climb into any finite pre-tree. This gives railyard labelings, which implies that at most S4 is valid. □

In general case, we need a parameter $n =$ the length of the sequence in $M$.

# No model of set theory has maximal $\Sigma_2$ theory

**Corollary**

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Take $\sigma(n) =$ "stage $n$ is successful."

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Take $\sigma(n) =$ "stage $n$ is successful."

**Corollary**

No $\omega$-standard model has a maximal $\Sigma_2$ theory.

# $\Sigma_2$ definability

**Theorem**

*In any countable model of set theory $M$, every element becomes $\Sigma_2$ definable from a natural number parameter in some top-extension of $M$.*

Indeed, there is a single definition and single parameter $n \in \omega^M$, such that every $a \in M$ is defined by that definition with that parameter in some top-extension $N$.

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**Proof.**

The definition is, "the unique set added at stage $n$," where $n$ is the first unsuccessful stage in $M$. □

# Parameter-free $\Sigma_2$ definability

## Corollary

For any countable $\omega$-standard model of set theory $M$, every $a \in M$ becomes $\Sigma_2$ definable without parameters in some top-extension $N$ of $M$.

Since $N$ is also $\omega$-standard, the result can be iterated.

# Maximality principle

It is not difficult to construct maximal consistent $\Sigma_2$ extensions of $\mathrm{ZFC}$.

**Theorem**

*Any model $M \models \mathrm{ZFC}$ with a maximal $\Sigma_2$ theory satisfies the top-extensional maximality principle, validating S5 for sentences.*

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- The modal language is capable of expressing sweeping general principles about the system.
- For many systems, one can identify the exact modal validities.
  - Lower bounds are found via structural features of the system.
  - Upper bounds are found by identifying presence of control statements.

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Thank you.

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